

Quantum Deformation Of Lattice Gauge Theory

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Abstract

A quantum deformation of 3-dimensional lattice gauge theory is defined by applying the Reshetikhin-Turaev functor to a Heegaard diagram associated to a given cell complex. In the root-of-unity case, the construction is carried out with a modular Hopf algebra. In the topological (weak-coupling) limit, the gauge theory partition function gives a 3-fold invariant, coinciding in the simplicial case with the Turaev-Viro one. We discuss bounded manifolds as well as links in manifolds. By a dimensional reduction, we obtain a q -deformed gauge theory on Riemann surfaces and find a connection with the algebraic Alekseev-Grosse-Schomerus approach.

1 Introduction

The lattice regularization of non-abelian gauge theory (LGT) proposed by K.Wilson in 1975 [1] plays a fundamental role of a non-perturbative definition of QCD. The main principle has been to give up the Poincaré invariance of the theory and preserve the local gauge symmetry as more fundamental. In the weak coupling regime, the broken translational and rotational invariance restores dynamically. If the gauge coupling is strictly equal to 0, the gauge strength tensor vanishes identically and the theory becomes topological: one can take any finite lattice from a given equivalence class without changing the content of the model.

The lattice formulation extends in a natural way the set of acceptable gauge groups for all compact groups while the continuous one is based on the notion of the Lie algebra thus excluding finite groups, for example. After the theory of quantum groups had appeared as a distinct mathematical subject [2, 3], the natural question arose whether the notion of gauge symmetry could be extended to incorporate quantum groups as well. This problem has not been only of academic interest. As was noticed in Ref. [4], the Ponzano-Regge model [5] (which coincides with the classical ($q \rightarrow 1$) limit of the Turaev-Viro construction [6]) can be represented as LGT defined on lattices dual to simplicial complexes. Then it was natural to assume that the Turaev-Viro invariant could be related to a kind of LGT built on quantum group symmetry (for the sake of brevity, we shall call it qQCD_3). This program was carried out in Ref. [7] explicitly (see also [8]). Technical difficulties originating from a complicated structure of the representation ring of $SL_q(2)$ at a root of unity, $q = e^{i\frac{2\pi}{k+2}}$, was avoided in [7] by establishing a direct connection with the ribbon graph invariants of Reshetikhin and Turaev [9]. The gauge invariance is implicit in this formulation (however, it does not mean that the model does not enjoy it). On the other hand, relative simplicity makes this explicit representation very convenient.

The classical ($q \rightarrow 1$) limit of the Turaev-Viro construction was discovered long ago by Ponzano and Regge in the framework of Regge calculus [5]. They argued that it can be regarded as a discretization of 3d gravity with the Einstein-Hilbert action. On the other hand, the Turaev-Viro invariant is related to Witten's Chern-Simons invariant [10]. Witten has shown that, with the $\text{ISO}(2,1)$ gauge group, the latter is connected with 3d quantum gravity. In the Euclidean regime for a negative cosmological constant, the gauge group becomes isomorphic to $SO(4) = SU(2) \times SU(2)$. It means that qQCD_3 possessing $SL_q(2)$ gauge symmetry is interesting from the physical point of view (an exposition of the subject can be found in Ref. [8]).

The structure of the answers for the partition function suggests that, in 2 dimensions, the corresponding q -deformed LGT (qQCD_2) is related to the topological $(G/G)_k$ coset model (as was argued in Ref. [11] these two models are in some sense dual to each other).

An alternative purely algebraic approach to qQCD_2 was put forward in Ref. [12, 13, 14]. The starting point for them was a Poisson bracket on 2d lattice connections proposed by

Fock and Rosly [15].

In 2 dimensions there is a cyclic order of links incident to a vertex. Demanding that variables performing gauge rotations at each vertex form a quasi-triangular Hopf algebra, Alekseev, Grosse and Schomerus have deduced an algebra of gauge fields. In contrast to this situation, in 3 dimensions there is a natural cyclic order of faces sharing the same link of a lattice, which suggests that one should start with gauge fields forming a Hopf algebra while gauge transformations are interpreted as changes of bases these fields act on. As in the quantum case there is no group manifold behind the construction, gauge fields and gauge transformations have clearly the different statuses. One could say that this is one more occurrence of the principle “Quantization removes degenerations”. Descending from 3 to 2 dimensions, one finds a model which seems, at first sight, to be different from the AGS one. However, as we shall show they are locally equivalent.

The outline of the paper is the following.

Chapter 2 is devoted to the construction of q -deformed LGT in three dimensions.

In Section 2.1 we introduce classical LGT and make some general remarks on its quantum deformation.

In Section 2.2 we collect facts from 3-manifold topology which are used in the sequel.

In Section 2.3 we introduce the Reshetikhin-Turaev functor in the form adopted for our purposes.

In Section 2.4 we define $q\text{QCD}_3$ functor and discuss the notion of gauge invariance within our framework.

In Section 2.5 we introduce the $q\text{QCD}_3$ partition function in the case of $\mathcal{U}_q(su(n))$ gauge group.

Section 2.6 is devoted to the root-of-unity case: $\mathcal{U}_q(sl(n, R))$, $q^\ell = 1$.

In Section 2.7 we prove that the weak-coupling partition function introduced in the previous section is a topological invariant.

In Section 2.8 we discuss the case of bounded manifolds and shortly outline the introduction of Wilson loop averages in our model.

Chapter 3 is devoted to the 2-dimensional case. Here we derive Verlinde’s formula and discuss a connection between our approach and the AGS algebra.

We conclude with a few general remarks.

2 $q\text{QCD}_3$

2.1 Formulation of the problem

To introduce lattice gauge theory, one needs a cell decomposition of a manifold (in physicist usage, a lattice). A gauge field is a map from a set of oriented edges to a compact group: $\ell \mapsto g_\ell \in G$. A change of an orientation corresponds to the conjugation: $g_\ell \rightarrow g_\ell^{-1}$.

One attaches to every vertex a G -module (usually the regular representation). Gauge

transformations rotate bases of the modules independently at each vertex. The gauge field is interpreted as performing a parallel transport between vertices, thus relating bases at adjacent ones.

If an oriented link, k , connects vertices v_2 and v_1 , the gauge transformation of the group element g_k is

$$g_k \rightarrow h_{v_1} g_k h_{v_2}^{-1} \quad (2.1)$$

A holonomy associated with a path $\{L\}$ in the lattice is an ordered product of gauge field elements along $\{L\}$:

$$h_L = \prod_{k \in L} g_k^{\epsilon_k} \quad (2.2)$$

where $\epsilon_k = +1$, if the k -th edge is directed along the path, and $\epsilon_k = -1$, if their directions are opposite.

Gauge invariant quantities are those taking values in the set of conjugacy classes of G . A trace of the holonomy along a closed loop in any representation of G is an example of such an invariant.

The Boltzmann weights are functions of holonomies along boundaries, ∂f , of faces, f . One of the standard choices is the so-called group heat kernel

$$W_\beta(h_{\partial f}) = \sum_R d_R \chi_R(h_{\partial f}) e^{-\beta C_R}, \quad (2.3)$$

In eq. (2.3), \sum_R is the sum over all finite dimensional irreps of a gauge group G ; $\chi_R(x)$ is the character of an irrep R ; $d_R = \chi_R(I)$ is its dimension; C_R is a second Casimir eigenvalue; β is a real parameter called a coupling constant. The construction makes sense for compact groups whose unitary finite dimensional irreps span the regular representation.

The choice (2.3) ensures that $W_\beta(h_{\partial f})$ becomes the group δ -function in the weak coupling limit, $\beta \rightarrow 0$: $W_0(h_{\partial f}) = \delta(h_{\partial f}, I)$. We shall call this limit *topological*.

The partition function is defined as the integral of the product of the Boltzmann weights over all faces:

$$\mathcal{Z}_\beta = \int_G \prod_\ell dg_\ell \prod_f W_\beta \left(\prod_{k \in \partial f} g_k^{\epsilon_k} \right) \quad (2.4)$$

where dg_ℓ is the Haar measure on the group G , and the product \prod_ℓ goes over all edges.

If the term “q-deformed” is to mean that gauge variables take values in a quantum group, any presentation of the model should be reducible to a form where the variables are represented in a standard fashion as matrices of non-commutative elements.

The simplest and most famous example is $SL_q(2)$, which can be introduced as the set of matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.5)$$

the entries of which obey the commutation relations

$$\begin{aligned} ba &= qab & db &= qbd & cb &= bc \\ ca &= qac & dc &= qcd & da - ad &= (q - q^{-1})bc \\ ad - q^{-1}bc &= 1 \end{aligned} \quad (2.6)$$

The relations (2.6) imply the existence of the R -matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad (2.7)$$

and the $RTT = TTR$ equation

$$(g \otimes 1)(1 \otimes g)R = R(1 \otimes g)(g \otimes 1) \quad (2.8)$$

The R -matrix obeys the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (2.9)$$

Indices show at which positions in the tensor cube of representation spaces, $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V}$, acts the R -matrix.

$SL_q(2)$ has two real forms: $SU_q(2)$, for real q , and $SL_q(2, R)$, for $|q| = 1$.

The matrices can be multiplied. If entries of both g and h obey Eq. (2.6) and are mutually commutative, the entries of the product gh obey (2.6) as well. Therefore, matrices on different links of a lattice have to co-commute with one another in the tensor product.

The algebra of matrices (2.5) naturally extends to the quasi-triangular Hopf algebra $\mathcal{F}_q(SL(2))$ of quantized functions on $SL(2)$. Owing to the famous duality, its basis is provided by the matrix elements of finite-dimensional irreducible representations of the quantized universal enveloping (QUE) algebra $\mathcal{U}_q(sl(2))$. Therefore, to construct qQCD, we have at hands co-multiplication, R -matrix, antipode and Clebsch-Gordan coefficients (CGC).

The q -deformation of Eq. (2.4) is roughly speaking a way to write it down in terms of elements of the Hopf algebra. *A priori*, it is not unique. The guiding principle here can be to identify any transformation of the construction with some isometry of a base cell complex in a self-consistent way. Then all algebraic manipulations become geometrically

meaningful. It is close in spirit to the Reshetikhin-Turaev functor from the category of ribbon tangles to the modular Hopf algebras [9]. Our presentation of qQCD_3 is in many respects inspired by their work. To describe it, we need to look at LGT from a bit more general than usual point of view.

2.2 Topological background

For reader's convenience we collect in this section some definitions which we shall use in the sequel.

A *k-cell* is a polyhedron homeomorphic to a k -dimensional ball.

A *cell complex* is a union of a finite number of cells such that an intersection of any 2 k -cells is either empty or a finite number of less dimensional cells.

A cell complex can be obtained starting with a finite set of points by attaching subsequently cells of higher dimensions, any cell being attached to a finite number of lower dimensional cells.

A union of all cells of dimension $\leq n$ is called an *n-skeleton*.

A cell complex is a *manifold* if and only if the neighbourhood of each vertex is a spherical ball.

A complex is called *simplicial* if all cells are simplexes (*i.e.*, points, links, triangles, tetrahedra, etc.).

Physicists usually mean by a *lattice* a cell complex such that an intersection of any two k -cells either empty or consists of only one entire less dimensional cell. We adopt this notion. Simplicial complexes are lattices by definition.

A *dual* complex, \tilde{C} , is constructed by putting into correspondence its k -cells to cells of C having complimentary dimensions, $n - k$.

To introduce LGT, we need a presentation of a cell complex, *i.e.*, an effective way to describe it unambiguously. In the classical case, one needs to know only a 2-skeleton of a complex.

From the topological point of view, the construction of LGT described at the beginning of the previous section is reminiscent of the definition of $H^1(C, G)$, the non-commutative first cohomology of C with coefficients in G . In the topological limit, all holonomies along contractible loops vanish and gauge fields obey the defining relations of $\pi_1(C)$. Therefore, being properly normalized, the partition function \mathcal{Z}_0 counts the number of conjugacy classes of injective homomorphisms from $\pi_1(C)$ into a gauge group G :

$$\mathcal{Z}_0 = |\text{Hom}(\pi_1(C), G)/G| \quad (2.10)$$

Of course, it makes sense only if G is finite. If G is a Lie group, one speaks about

a *moduli space of flat G -connections*, which is defined as a set of fields modulo gauge transformations:

$$\mathcal{M}_G := \{Hom(\pi_1(C), G)/G\} \quad (2.11)$$

It is easy to see that classical topological LGT is completely determined by a homotopy type of a complex. The construction of qQCD₃ requires a more precise presentation of a complex.

It is known that any oriented 3-manifold can be obtained by gluing up two 3-dimensional handlebodies along their boundaries. This operation is the *Heegaard splitting*. The minimal genus of the handlebodies is called the *Heegaard genus* of the manifold.

We can obtain a Heegaard splitting for a given oriented manifold M from its cellular decomposition, C , as follows. We take a tubular neighbourhood, H , of the 1-skeleton of C . The complement of H in M , $\widetilde{H} = M \setminus H$, can be regarded as a tubular neighbourhood of the 1-skeleton of the dual complex \widetilde{C} .

Every 1-cell $\sigma_i^1 \in C$ determines a disk $D_i \subset H$ whose detachment destroys a handle of H . The boundaries of the disks $\partial D_i \subset \partial H$ give a system of cycles on the boundary, ∂H , of the handlebody H . We shall call them the α -cycles: $\alpha_i := \partial D_i$. Dual 1-cells $\widetilde{\sigma}_j^1 \in \widetilde{C}$ determine analogously a system of $\tilde{\alpha}$ -cycles on the boundary, $\partial \widetilde{H}$, of \widetilde{H} . Images of the $\tilde{\alpha}$ -cycles on ∂H produced by a gluing homomorphism h are called the *characteristic curves* (or γ -cycles) of the Heegaard diagram and define the manifold $M = H \cup_h \widetilde{H}$ unambiguously.

Let us fix a number of the disks $\{\widehat{D}\} \subset \{D\}$ such that the detachment of them makes the handlebody connected and simply-connected (*i.e.*, $H \setminus \{\widehat{D}\} \cong B^3$). We can put into correspondence a generator a_i of the fundamental group to each disk $D_i \in \{\widehat{D}\}$. Defining relations are read off in an obvious way from a system of the characteristic curves $\{\gamma\}$. That is, if γ_j intersects disks $D_{j_1}, D_{j_2}, \dots, D_{j_k}$ subsequently, then the corresponding relator is $\Gamma_j = a_{j_1}^{\epsilon_{j_1}} a_{j_2}^{\epsilon_{j_2}} \dots a_{j_k}^{\epsilon_{j_k}}$, where $\epsilon_k = \pm 1$ is the intersection number depending on a mutual orientation of γ_j and the k -th disk at the intersection point. This set of relators is of course excessive. A minimal set can be fixed by choosing a number of $\tilde{\alpha}$ -cycles which span a disjoint collection of disks $\{\widehat{\widetilde{D}}\}$ in the complementary handlebody $\widetilde{H} = M \setminus H$ such that the detachment of all the disks from the set $\{\widehat{\widetilde{D}}\}$ makes \widetilde{H} a 3-ball: $\widetilde{H} \setminus \{\widehat{\widetilde{D}}\} \cong B^3$.

One can deform a Heegaard diagram by any 2-dimensional isomorphism of a boundary ∂H which extends to the whole handlebody H . A set of generators for such isomorphisms is called in the literature the Suzuki moves (see, *e.g.*, Refs. [16, 17] for an exposition accessible to a physicist).

It can be shown that any class of isotopic diffeomorphisms of a genus g surface M_g^2 onto itself has a representative which can be constructed as a composition of the Dehn twists, T_μ^ϵ , where μ is one of the basic cycles on M_g^2 and $\epsilon = \pm 1$. One detaches from M_g^2 a thin neighborhood $U_\mu \cong S^1 \times [0, 1]$ of a cycle μ and then attaches it back after the full

twist $U_\mu \rightarrow U_\mu: \varphi \times t \rightarrow (\varphi + 2\pi\epsilon t) \times t$ (where $t \in [0, 1]$ and $\phi \in [0, 2\pi]$ parametrizes S^1).

In the sequel, we shall only need the following fact: all the Suzuki moves are combinations of Dehn twists on loops in ∂H which bound disks $D \subset H$, except for the *handle slide* defined in the following way. Imagine solid handles attached to a surface of a spherical ball. One drags one end of a handle up, along and down another handle. As a result of this operation, an α -cycle corresponding to the second handle slides around an α -cycle corresponding to the first one. It can be described as a multiplication of loops on ∂H defined in the standard fashion as in the definition of the fundamental group $\pi_1(M_g^2)$

The same operations can be applied to \widetilde{H} as well.

It is a classical result that any two Heegaard diagrams representing the same manifold can be connected by a sequence of the following operations:

1. Dehn twists on loops contractible in H or \widetilde{H} . They do not change a presentation of $\pi_1(C)$.
2. *Cycle slide*, which consists in the multiplication of a cycle by another one: $\gamma_j \rightarrow \gamma_j \gamma_k$. It means that a relator Γ_j in a presentation of $\pi_1(C)$ is substituted by $\Gamma_j \Gamma_k$. The same operation applied to the α -cycles, $\alpha_j \rightarrow \alpha_j \alpha_k$, corresponds to the change of generators of $\pi_1(C)$: a_j is substituted for $a_j a_k$.
3. *Stabilization*, which consists in adding a new handle to H and extending a gluing diffeomorphism by the identity on its boundary. It means that one adds one characteristic curve and one α -cycle to a Heegaard diagram or, equivalently, a new generator a_{g+1} along with the trivial relation, $\Gamma_{g+1} = a_{g+1} = 1$, to a presentation of $\pi_1(C)$.

It should be noted that an isotopy within a handlebody itself cannot necessarily take place for its embeddings in R^3 . The obvious obstruction is that the characteristic curves can become linked in R^3 .

We shall need the operation of the *connected sum* of two manifolds: $M = M_1 \# M_2$. One constructs M by deleting spherical balls from M_1 and M_2 and then gluing the manifolds together along the boundaries. Obviously, $M \# S^3 \cong M$, which can be represented as the attachment of a single 3-cell to the spherical boundary of the ball obtained from M . This operation introduces an abelian semi-group structure and any 3-fold invariant can be regarded as a representation of this semi-group.

A manifold is called *simple*, if it cannot be represented as a connected sum of two non-spherical manifolds. Any compact oriented 3-manifold possesses a unique expansion into a connected sum of simple manifolds.

By performing a Heegaard splitting, a manifold is constructed out of two handlebodies joined by some homeomorphism of their boundaries. Such a homeomorphism can be continued into small neighborhoods of the boundaries. It means that the characteristic

curves can submerge a bit into the inside of H . If $n_{ij} = (\alpha_i, \gamma_j)$ is an intersection number of two cycles on the boundary, then γ_j will have after the deformation the same number as a linking coefficient with α_i . The *linking coefficient* of two loops in R^3 is equal, by definition, to the intersection number of the first with a disk spanned by the second.

If α_i and γ_j are linked, the corresponding 1-cell, σ_i^1 , enters the boundary of the corresponding 2-cell, σ_j^2 : $\sigma_i^1 \in \partial\sigma_j^2$. And vice versa for co-boundaries: $\sigma_j^2 \in \delta\sigma_i^1$. The boundary of a 2-cell defines a natural cyclic order of 1-cells belonging to it. A peculiarity of the dimension 3 is that 2-cells forming a co-boundary of a 1-cell are naturally ordered as well. It is a cyclic order of dual 1-cells forming a boundary of a dual 2-cell.

2.3 Reshetikhin-Turaev functor

The quantized function (QF) algebra $\mathcal{F}_q(SL(n))$ is dual to the QUE algebra $\mathcal{U}_q(sl(n))$, therefore the topological basis of $\mathcal{F}_q(SL(n))$ is given by the matrix elements of irreducible representations of $\mathcal{U}_q(sl(n))$. For example, the 2×2 matrix realization of $SL_q(2)$ given in Eqs. (2.5) and (2.6) exactly corresponds to the matrix elements of the 2-dimensional irreducible representation of $\mathcal{U}_q(sl(2))$. In this paper, we shall deal with real forms of $\mathcal{U}_q(sl(n, C))$ with respect to some $*$ -structures, whose existence is always assumed.

The discussion in the previous section suggests that we can construct qQCD_3 with help of the Reshetikhin-Turaev functor from the category of colored ribbon tangles **ctang** to the category of representation rings of $\mathcal{U}_q(sl(n))$, $\mathbf{rep}_{\mathcal{U}_q}$.

The basic geometric object is a *tangle*, which was defined in Ref. [9] as “a link of circles and segments in the 3-ball, where it is assumed that ends of segments lie on the boundary of the ball”. One puts into correspondence to every tangle a linear operator, f , acting on a tensor product of modules associated with segments (which have therefore to be oriented).

$$f : V_{i_1} \otimes \dots \otimes V_{i_n} \rightarrow V_{j_1} \otimes \dots \otimes V_{j_k} \quad (2.12)$$

or, graphically,

$$f_{i_1 \dots i_n}^{j_1 \dots j_k} \cong \begin{array}{c} \begin{array}{c} j_1 \quad j_k \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \boxed{f} \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ i_1 \quad i_n \end{array} \end{array} \quad (2.13)$$

where j_1, \dots, j_k and i_1, \dots, i_n are some indices numerating modules. The simplest example

is the identity operator represented by a single segment:

$$\delta_{\alpha,\beta} \cong \begin{array}{c} \alpha \\ \uparrow \\ \downarrow \\ \beta \end{array} \quad (2.14)$$

All modules considered in this paper are assumed to be irreducible. We shall draw linear operators acting on them as small boxes (coupons) with labels inside. The elementary building blocks are the matrix elements

$$D_{\alpha,\beta}^i(a) \cong \begin{array}{c} \alpha \\ \uparrow \\ \boxed{a} \\ \uparrow \\ i, \beta \end{array} \quad (2.15)$$

where a is an element of \mathcal{U}_q . The arrows show a direction of the action of an operator. We use the Greek letters to numerate basis vectors of irreducible modules and the Latin ones, to numerate the modules. They will often be omitted.

\mathcal{U}_q possesses several $*$ -structures. We shall draw conjugate objects as

$$D_{\alpha,\beta}^i(a^*) \cong \begin{array}{c} \beta \\ \downarrow \\ \boxed{a} \\ \downarrow \\ i, \alpha \end{array} = \begin{array}{c} \overline{\beta} \\ \uparrow \\ \boxed{a} \\ \uparrow \\ i, \overline{\alpha} \end{array} \quad (2.16)$$

The last equality takes place for a real form of \mathcal{U}_q , where the $*$ -structure matches basis vectors of a module: $\alpha \rightarrow \overline{\alpha}$.

The operators form an *algebra* \mathcal{A} . We can translate this property in pictures as

$$\left\{ a, b \in \mathcal{A} \rightarrow ab \in \mathcal{A} \right\} \cong \left\{ \begin{array}{c} \uparrow \\ \boxed{a} \\ \uparrow \\ \boxed{b} \\ \uparrow \end{array} \right\} = \left\{ \begin{array}{c} \uparrow \\ \boxed{ab} \\ \uparrow \end{array} \right\} \quad (2.17)$$

$$\left\{ \exists 1 \in \mathcal{A} : a1 = 1a = a, \forall a \in \mathcal{A} \right\} \cong \left\{ \begin{array}{c} \uparrow \\ \boxed{1a} \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \boxed{a1} \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \boxed{a} \\ \uparrow \end{array} \right\} \quad (2.18)$$

\mathcal{A} is a ring, *i.e.*, an Abelian group under some $+$ operation, which we shall understand as a formal sum of pictures with the natural definition of the multiplication by an integer number.

To have a *bi-algebra* structure on \mathcal{A} , we need a *co-multiplication* $\Delta : V \rightarrow V \otimes V$ and a *co-unit* ε . We introduce Δ as

$$\Delta(a) \cong \begin{array}{c} \uparrow \uparrow \\ \boxed{\Delta(a)} \\ \uparrow \uparrow \end{array} = \sum_i \begin{array}{c} \begin{array}{c} \diagup \quad \diagdown \\ \uparrow \\ \boxed{a} \\ \downarrow \end{array} \\ \begin{array}{c} \downarrow \quad \downarrow \\ \uparrow \end{array} \end{array} = \begin{array}{c} \uparrow \quad \uparrow \\ \boxed{a} \quad \boxed{a} \\ \uparrow \quad \uparrow \end{array} \quad (2.19)$$

In general, the last equality is simply a convenient pictorial representation and has to be given a precise meaning in every particular case.

In Eq. (2.19), the 3-valent vertices,

$$\begin{array}{c} j_3, \alpha_3 \\ | \\ \diagup \quad \diagdown \\ j_1, \alpha_1 \quad j_2, \alpha_2 \end{array} \cong C_{j_1 \alpha_1; j_2 \alpha_2}^{j_3 \alpha_3} \quad \begin{array}{c} j_1, \alpha_1 \quad j_2, \alpha_2 \\ \diagdown \quad \diagup \\ | \\ j_3, \alpha_3 \end{array} \cong \overline{C}_{j_3 \alpha_3}^{j_1 \alpha_1; j_2 \alpha_2} \quad (2.20)$$

are the quantum Clebsch-Gordan coefficients $e_{\alpha_1}^{j_1} \otimes e_{\alpha_2}^{j_2} = \sum_{j_3, \alpha_3} C_{j_1 \alpha_1; j_2 \alpha_2}^{j_3 \alpha_3} e_{\alpha_3}^{j_3}$ (e_{α}^j is a basis of V_j). They obey the properties

$$\sum_i \begin{array}{c} \begin{array}{c} \diagup \quad \diagdown \\ \uparrow \\ \boxed{a} \\ \downarrow \end{array} \\ \begin{array}{c} \downarrow \quad \downarrow \\ \uparrow \end{array} \end{array} = \begin{array}{c} \uparrow \quad \uparrow \\ \boxed{a} \quad \boxed{a} \\ \uparrow \quad \uparrow \end{array} ; \quad \begin{array}{c} \uparrow \\ \bigcirc \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \boxed{a} \\ \uparrow \end{array} \quad (2.21)$$

which simply means that they are elements of a unitary matrix connecting bases in V and $V \otimes V$:

$$\sum_{i,\beta} C_{j_1\alpha_1;j_2\alpha_2}^{i\beta} \overline{C}_{i\beta}^{j'_1\alpha'_1;j'_2\alpha'_2} = \delta_{j_1,j'_1} \delta_{j_2,j'_2} \delta_{\alpha_1,\alpha'_1} \delta_{\alpha_2,\alpha'_2}$$

We can check the properties of the co-multiplication graphically

$$\left\{ \Delta(ab) = \Delta(a)\Delta(b) \right\} \cong \left\{ \begin{array}{c} \text{Y-junction} \\ \uparrow \\ \boxed{ab} \\ \uparrow \\ \text{Y-junction} \end{array} = \begin{array}{c} \text{Y-junction} \\ \uparrow \\ \boxed{a} \\ \uparrow \\ \boxed{b} \\ \uparrow \\ \text{Y-junction} \end{array} = \begin{array}{c} \text{Y-junction} \\ \uparrow \\ \boxed{a} \\ \uparrow \\ \bigcirc \\ \uparrow \\ \boxed{b} \\ \uparrow \\ \text{Y-junction} \end{array} \right\} \quad (2.22)$$

and

$$\left\{ \Delta(1) = 1 \otimes 1 \right\} \cong \left\{ \begin{array}{c} \text{Y-junction} \\ \uparrow \\ \text{Y-junction} \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} \right\} \quad (2.23)$$

In these formulas, the sum over intermediate states is assumed. In what follows, we shall often omit the sum sign in pictures.

Thus, the co-associativity is coded in the properties of the Clebsch-Gordan coefficients.

The co-unit is a homomorphism to an abelian group associated with a field over which \mathcal{A} is defined.

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \quad \varepsilon(1) = 1 \quad (2.24)$$

We shall connect the co-unit with a projection on the trivial representation of a quantum group. In other words, with the group integration.

To have the *Hopf algebra* structure on \mathcal{A} , we introduce an *antipode* map: $S : \mathcal{A} \rightarrow \mathcal{A}$:

$$S(\boxed{a}) = \begin{array}{c} \uparrow \\ \boxed{a} \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \boxed{a} \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \boxed{a^*} \\ \uparrow \end{array} \quad (2.25)$$

obeying

$$\cdot (S \otimes id) \circ \Delta = \cdot (id \otimes S) \circ \Delta = 1 \circ \varepsilon \quad (2.26)$$

which looks graphically as

$$\cdot (S \otimes id) \circ \Delta = \text{[Diagram: A box labeled } a \text{ with two incoming lines from below and two outgoing lines from above that cross and then go up]} = \text{[Diagram: A box labeled } a \text{ with two incoming lines from below, one of which loops around to the left and then goes up]} = \text{[Diagram: A single vertical line with an upward arrow]} \varepsilon(a) \quad (2.27)$$

where a is an arbitrary element of \mathcal{A} and \circ means a composition of operations. This property shows that the antipode can serve as a q -analog of the inverse. However, in general, $S^2 \neq 1$.

The maps $V \otimes V \rightarrow C$ and $C \rightarrow V \otimes V$ are constructed with help of CGC:

$$\text{[Diagram: A loop with two vertical lines labeled } j \text{]} := \sqrt{d_j} \text{[Diagram: A Y-junction with two lines labeled } j \text{ going down and one line labeled } 0 \text{ going up]} ; \quad \text{[Diagram: A loop with two vertical lines labeled } j \text{]} := \sqrt{d_j} \text{[Diagram: A Y-junction with two lines labeled } j \text{ going up and one line labeled } 0 \text{ going down]} \quad (2.28)$$

where d_j is the quantum dimension of the module V_j . These objects become the ordinary δ -functions in the $q \rightarrow 1$ limit.

The self-consistency requires that

$$\{S(ab) = S(a)S(b)\} \cong \left\{ \text{[Diagram: A box labeled } ab \text{ with a loop around it]} = \text{[Diagram: Two boxes labeled } b \text{ and } a \text{ stacked vertically with a loop around them]} = \text{[Diagram: Two boxes labeled } a \text{ and } b \text{ side-by-side with loops around each]} \right\} \quad (2.29)$$

For this property to hold, it is important that

$$\begin{array}{c} \uparrow \\ \boxed{a} \\ \boxed{b} \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \boxed{a} \\ \text{---} \boxed{b} \\ \uparrow \end{array} = \begin{array}{c} \text{---} \boxed{b} \\ \boxed{a} \\ \uparrow \end{array} \quad (2.30)$$

The other property of the antipode is $\Delta \circ Sa = \tau \circ (S \otimes S) \circ \Delta a$. It can be checked graphically:

$$\Delta \circ \begin{array}{c} \uparrow \\ \boxed{a} \\ \uparrow \end{array} = \begin{array}{c} \text{---} \boxed{a} \\ \uparrow \end{array} = \begin{array}{c} \text{---} \boxed{a} \\ \uparrow \end{array} = \tau \circ (S \otimes S) \begin{array}{c} \text{---} \boxed{a} \\ \uparrow \end{array} \quad (2.31)$$

where τ is the flip operator: $V_{j_1} \otimes V_{j_2} \xrightarrow{\tau} V_{j_2} \otimes V_{j_1}$.

Having equipped \mathcal{A} with an R -matrix, we obtain a quasi-triangular Hopf algebra (\mathcal{A}, R) . The R -matrix obeys the Yang-Baxter equation (2.9). In our context, it will be more convenient to consider the \check{R} -matrix

$$\check{R} := \tau \circ R \quad (2.32)$$

which can be represented graphically as

$$\left\{ \check{R} = \tau \circ \sum_i \alpha_i \otimes \beta_i \right\} \cong \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad (2.33)$$

It is invertible

$$\left\{ \check{R}^{-1} = \tau \circ \sum_i \beta_i \otimes S(\alpha_i) \right\} \cong \left\{ \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right\} \quad (2.34)$$

A Hopf algebra is called *triangular* if $\check{R}^2 = 1$.

The standard definition of the \check{R} -matrix requires that the following properties hold

$$\left\{ (\Delta \otimes id) \check{R} = \check{R}_{12} \check{R}_{23} \right\} \cong \left\{ (\Delta \otimes id) \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\} \quad (2.35)$$

$$\left\{ (id \otimes \Delta) \check{R} = \check{R}_{12} \check{R}_{23} \right\} \cong \left\{ (id \otimes \Delta) \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\} \quad (2.36)$$

along with the general form of Eq. (2.8)

$$\left\{ \check{R}(\Delta a) \check{R}^{-1} = \Delta a, \forall a \in \mathcal{A} \right\} \cong \left\{ \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \begin{array}{|c|c|} \hline a & a \\ \hline \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \begin{array}{|c|c|} \hline a & a \\ \hline \end{array} \right\} \quad (2.37)$$

These equations are equivalent to the Yang-Baxter one

$$\left\{ \check{R}_{12} \check{R}_{23} \check{R}_{12} = \check{R}_{23} \check{R}_{12} \check{R}_{23} \right\} \cong \left\{ \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\} \quad (2.38)$$

The standard relations including the antipode are

$$\left\{ (S \otimes id) \check{R} = \check{R}^{-1} \right\} \cong \left\{ (S \otimes id) \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\} \quad (2.39)$$

$$\left\{ (id \otimes S) \check{R}^{-1} = \check{R} \right\} \cong \left\{ (id \otimes S) \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\} \quad (2.40)$$

and those with the co-unit are


$$\left\{ (\varepsilon \otimes id) \check{R} = (id \otimes \varepsilon) \check{R} = 1 \right\} \cong \left\{ \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} | \\ | \\ | \end{array} \right\} \quad (2.41)$$

The standard way to introduce the ribbon Hopf algebra structure on \mathcal{A} is to bring forward the element $u \in \mathcal{A}$ defined as [9]

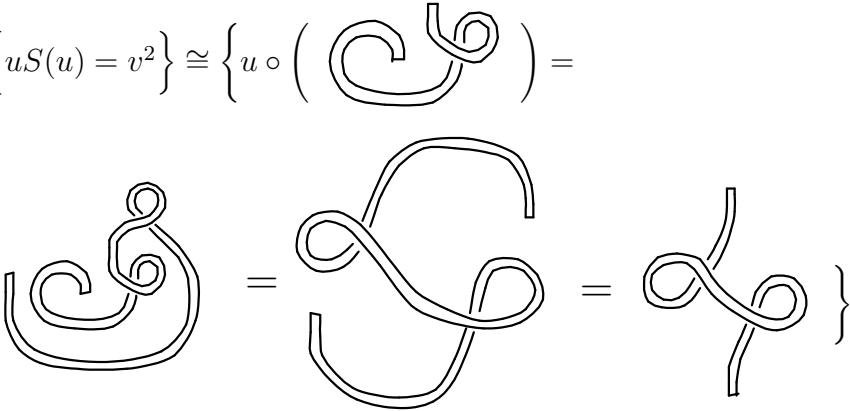
$$\left\{ u := \sum_i S(\beta_i) \alpha_i \right\} \cong \text{diagram} \quad (2.42)$$

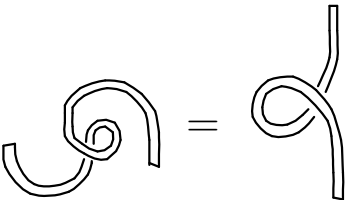

The element $v^2 = uS(u)$ lies in the center of \mathcal{A} .

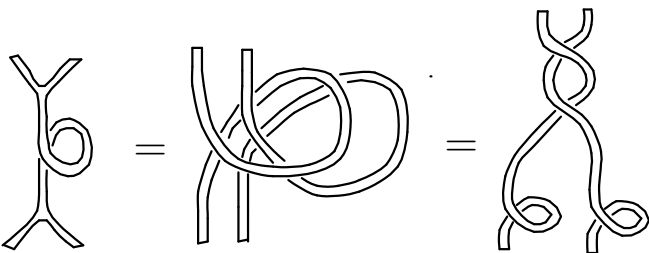
A ribbon Hopf algebra $\mathcal{U}_q = (\mathcal{A}, \check{R}, v)$ is a quasi-triangular Hopf algebra (\mathcal{A}, \check{R}) equipped with a central invertible element $v \in \mathcal{A}$,

$$v := \text{diagram} = \text{diagram} \quad v^{-1} := \text{diagram} = \text{diagram} \quad (2.43)$$


obeying the properties: $\varepsilon(v) = 1$,

$$\left\{ uS(u) = v^2 \right\} \cong \left\{ u \circ \left(\text{diagram} \right) = \right. \\ \left. \text{diagram} = \text{diagram} = \text{diagram} \right\} \quad (2.44)$$


$$\left\{ S(v) = v \right\} \cong \left\{ \text{diagram} = \text{diagram} \right\} \quad (2.45)$$


$$\left\{ \Delta(v) = (\check{R}^2)^{-1}(v \otimes v) \right\} \cong \left\{ \text{diagram} = \text{diagram} = \text{diagram} \right\} \quad (2.46)$$


The element

$$uv^{-1} \cong \text{[diagram of two loops with a twist]} = \text{[diagram of two parallel loops]} \quad (2.47)$$

allows for defining the q-trace of an operator

$$\text{qtr}(a) := \text{tr}(auv^{-1}) \cong \text{[diagram of a loop with a box labeled } a \text{]} \quad (2.48)$$

In the tensor square of spaces it takes the form

$$\left\{ \text{tr}[a \otimes b \circ \Delta(uv^{-1})] = \text{qtr}(a) \text{qtr}(b) \right\} \cong \left\{ \text{[diagram of two boxes } a \text{ and } b \text{ in a single loop]} = \text{[diagram of two separate loops, one with } a \text{ and one with } b \text{]} \right\} \quad (2.49)$$

Following Reshetikhin and Turaev, we shall call this operation the *closing* of a tangle.

The quantum dimension of a module, V_j , is, by definition, the q-trace of the identity operator:

$$d_j := \text{qtr}(1_{V_j}) \cong \text{[diagram of an empty loop]} \quad (2.50)$$

2.4 Algebra of fields and gauge invariance

In Section 2.2 we have described a Heegaard diagram as a handlebody with a given system of α -cycles and characteristic curves on its boundary. Every α -cycle span a disk D in a handlebody H . The disk can be thickened to a plate P . In this way we obtain a collection of disjoint plates in H ($P_i \cap P_j = \emptyset$, if $i \neq j$). Each plate corresponds to a 1-cell of a base cell complex C from which the Heegaard diagram has been read off. By detaching

the plates, H reassembles into a collection of 3-balls $\{B\}$, each corresponding to a 0-cell of C .

DEFINITION 1: We construct qQCD_3 functor in the following way:

1. A gauge variable taking values in a ribbon QUE algebra \mathcal{U}_q is put into correspondence to each plate:

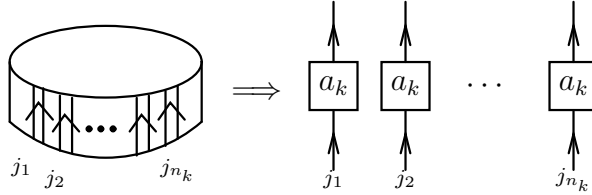
$$P_k \longrightarrow a_k \in \mathcal{U}_q$$

The variables attached to different plates are distinct elements of \mathcal{U}_q , hence their matrix elements are co-commutative.

2. All the characteristic curves are colored with irreducible finite-dimensional representations of \mathcal{U}_q .
3. If on a boundary of the k -th plate there are n_k disjoint cuts of the characteristic curves colored with representations j_1, j_2, \dots, j_{n_k} , we construct a *gauge field tangle* by repeatedly applying the co-multiplication:

$$F_k = \Delta^{n_k-1}(a_k) : V_{j_1} \otimes \dots \otimes V_{j_{n_k}} \rightarrow V_{j_1} \otimes \dots \otimes V_{j_{n_k}}$$

or, graphically,



One has to respect the cyclic order and mutual orientations of the cuts. A reversion of an orientation of a cut corresponds to the conjugation^(†) of the corresponding matrix element.

4. One puts into correspondence to each ball $B_i \in \{B\}$ carrying a pattern of the characteristic curves on its boundary a *vertex tangle* by using the Reshetikhin-Turaev functor $\mathbf{ctang} \rightarrow \mathbf{rep}_{\mathcal{U}_q}$.
5. In the end, the pieces are attached together. For it, one embeds the handlebody into R^3 in such a way that the cuts of the characteristic curves on boundaries of the plates project to distinct points on the (x, y) plane and onto disjoint segments on the (x, z) plane. Then one can use the (x, z) projection of the vertex tangles to complete the construction in terms of elements of a ribbon Hopf algebra as was depicted in the previous section. A result is a functional taking values in C . \square

^{†)} with respect to some fixed $*$ -structure.

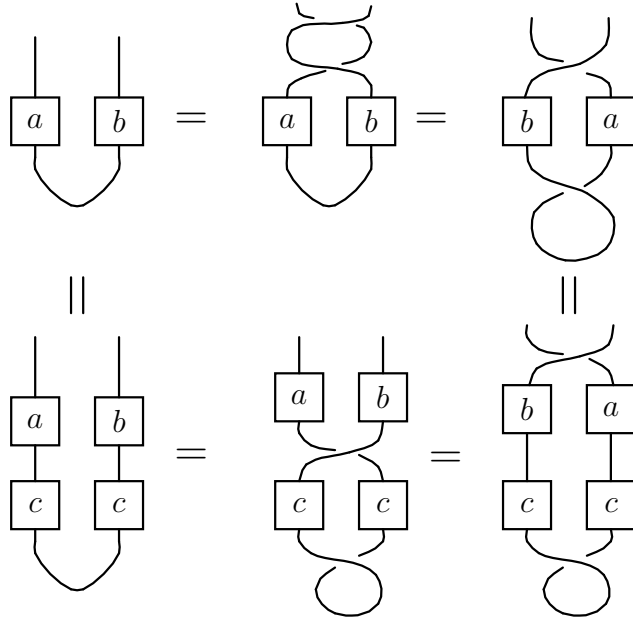


Figure 1: An illustration of the non-cocommutativity of fields adjacent via a vertex.

Remarks: 1) Let us notice that the initial data are colorings and directions of the characteristic curves as well as \mathcal{U}_q elements attached to the plates.

2) One can sum over all the colorings with arbitrary weights as in Eq. (2.4). As we are restricted to real forms of $\mathcal{U}_q(sl(n))$, the result has to be independent of the directions of the characteristic curves.

3) Modules appearing in different vertex tangles are independent. However, after the assemblage, all pieces are fit together and, permuting matrix elements adjacent via a vertex tangle, one has to deform the tangle, which means some effective non-cocommutativity. We shall dwell at this point later. Now, let us simply illustrate what may happen by the example shown in Figure 1.

4) Clearly, the construction gives different results for non-isotopic embeddings of H into R^3 . This lack of self-consistency will disappear after the integration over gauge fields (see the next section).

Let us now discuss the issue of gauge invariance within our framework. In the general settings of gauge theory the basic object is a fiber bundle over a base manifold. Gauge field performs a parallel transport of fibers and thus is interpreted as a G -connection in sections of the bundle. Gauge transformations act by automorphisms of the fibers. To make it explicit, one has to choose some G -basis at each point of the base. A quantity is gauge invariant if it is independent of a particular choice of the bases.

In LGT a base manifold is substituted by a finite cell complex. Therefore, instead of a fiber bundle, one has a tensor product of G -modules, one for each 0-cell in the complex.

2.5 Partition function

$$\left\{ \int da \, D^j(a) = \delta_{j,0} \right\} \cong \left\{ \varepsilon(\underset{j}{\overline{\square}}) = \delta_{j,0} \right\} \quad (2.51)$$
$$\varepsilon(\begin{array}{c} \beta_1 \quad \beta_2 \\ \boxed{a} \quad \boxed{a} \\ j_1 \alpha_1 \quad j_2 \alpha_2 \end{array}) = \sum \varepsilon(\begin{array}{c} \beta_1 \quad \beta_2 \\ \boxed{a} \\ j_1 \alpha_1 \quad j_2 \alpha_2 \end{array}) = \frac{\delta_{j_1, j_2}}{d_{j_1}} \begin{array}{c} \beta_1 \quad \beta_2 \\ \text{---} \text{---} \\ j_1 \alpha_1 \quad j_2 \alpha_2 \end{array} \quad (2.52)$$

The selfconsistency of the definition can be easily checked

$$\begin{array}{c} \varepsilon(\boxed{a} \quad \boxed{a}) \\ \varepsilon(\boxed{b} \quad \boxed{b}) \end{array} = \frac{1}{d_j^2} \begin{array}{c} \cup \\ \bigcirc \\ \cup \end{array} = \frac{1}{d_j} \begin{array}{c} \cup \\ \cup \end{array} \quad (2.53)$$

The main property of the Haar integral is the right/left invariance:

$$\varepsilon\left(\begin{array}{c} | \\ \boxed{abc} \\ | \end{array} \begin{array}{c} | \\ \boxed{abc} \\ | \end{array}\right) = \sum \varepsilon\left(\begin{array}{c} \boxed{a} \quad \boxed{a} \\ | \quad | \\ \text{---} \\ \boxed{b} \\ | \\ \boxed{c} \quad \boxed{c} \\ | \quad | \end{array}\right) = \frac{1}{d_j} \begin{array}{c} \boxed{a} \quad \boxed{a} \\ | \quad | \\ \text{---} \\ \boxed{c} \quad \boxed{c} \\ | \quad | \end{array} = \frac{1}{d_j} \begin{array}{c} \cup \\ \cup \end{array} \quad (2.54)$$

In the general case, the invariance easily follows from the properties of the Clebsch-Gordan coefficients and the antipode.

The integral should be used with some caution. For example, a reader has to be aware that

$$\varepsilon\left(\begin{array}{c} | \\ \boxed{a} \\ | \end{array}\right) \Bigg| \neq \varepsilon\left(\begin{array}{c} | \\ \boxed{a} \\ | \end{array}\right) \quad (2.55)$$

and the r.h.s. of this formula makes no sense. Otherwise, one could easily arrive at contradictions. Expressions like $\int dx dy f(x, y)$ are inadequate in the quantum case. To exclude ambiguity, we shall always connect the integration with the linear operator acting on a tensor product of modules constructed with help of the co-multiplication:

DEFINITION 2:

$$\int := \varepsilon\left(\Delta^n(\quad)\right) : (V_1 \otimes \dots \otimes V_{n+1} \xrightarrow{\Delta^n} V_1 \otimes \dots \otimes V_{n+1}) \xrightarrow{\varepsilon} C$$

It can be calculated recursively:

$$\varepsilon\left(\begin{array}{c} | \\ \boxed{a} \cdots \boxed{a} \boxed{a} \\ | \quad | \quad | \end{array}\right) = \sum \varepsilon\left(\begin{array}{c} | \\ \boxed{a} \cdots \boxed{a} \\ | \quad | \quad \text{---} \end{array}\right) \quad (2.56)$$

To complete our construction we need to specify a real form of $\mathcal{U}_q(sl(n))$ with respect to some fixed $*$ -structure. We are interested in 2 cases: (i) $\mathcal{U}_q(su(n))$ which makes sense for real q and (ii) $\mathcal{U}_q(sl(n, R))$ for $|q| = 1$. The Hopf $*$ -algebra $\mathcal{U}_q(su(n))$ has been well investigated starting from the pioneering works of Woronowicz and Vaksman and Soibelman [3]. We need the following facts:

1. There is the one-to-one correspondence between finite dimensional irreducible representations of $\mathcal{U}_q(su(n))$ and the classical algebra $\mathcal{U}(su(n))$.
2. The representation ring of $\mathcal{U}_q(su(n))$ spanned by matrix elements of finite-dimensional irreducible representations can be regarded as the q -deformation of the algebra of regular functions on $SU(N)$ (the quantum Peter-Weyl theorem). In particular, there exists a q -analog of the group δ -function.

3. There exists a q -analog of the Haar measure. The matrix elements are orthogonal with respect to it. An explicit representation of them can be given in terms of q -special functions. In this case the group integration is performed with help of the so-called Jackson integral from the q -special function theory.

Now, we are in a position to define the $q\text{QCD}_3$ partition function. We shall denote the number of k -cells in a complex as N_k .

DEFINITION 3: We take the construction of the $q\text{QCD}_3$ functor introduced in the previous section. Then

1. We color the characteristic curves with $\mathcal{U}_q(su(n))$ irreps: $\gamma_i \rightarrow j_i$, $i = 1, \dots, N_2$.
2. We put into correspondence to every plate P_k ($k = 1, \dots, N_1$) the integral

$$P_k \longrightarrow \varepsilon(\Delta^{n_k-1}(a_k)), \quad a_k \in \mathcal{U}_q(su(n))$$

3. By applying Eqs. (2.51), (2.52) and (2.56) we obtain a collection of closed 3-valent ribbon graphs $\{\tau\}$, the number of which equals the number of 0-cells in a base cell complex. By using the Reshetikhin-Turaev functor, we calculate the quantum invariant^(‡), $J(\tau_k)$, for each connected component, τ_k . Let us denote their product as

$$Z_{j_1 \dots j_{N_2}} = \prod_{k=1}^{N_0} J(\tau_k) \quad (2.57)$$

4. The partition function equals the sum over all colorings of the characteristic curves

$$\mathcal{Z}_\beta = \sum_{\{j_1 \dots j_{N_2}\}} \prod_{k=1}^{N_2} (d_{j_k} e^{-\beta C_{j_k}}) Z_{j_1 \dots j_{N_2}} \quad (2.58)$$

where d_j is the quantum dimension and C_j is a second Casimir eigenvalue.

Remarks: 1) If $q = 1$, this definition reduces to the one given in Eqs. (2.3) and (2.4).
 2) \mathcal{Z}_β is a gauge invariant quantity in the sense described in the previous section. Indeed, as any vertex tangle after the integration gives a closed ribbon 3-valent graph, the choice of a basis attached to it is irrelevant.
 3) After the integration, all non-isotopic embeddings of a handlebody H into R^3 become equivalent and the consideration can be restricted to isotopies of the handlebody itself.
 4) If one considers a cell complex dual to a simplicial one, the ribbon graph invariants $J(\tau_k)$ in Eq. (2.57) coincide with the quantum 6- j symbols in the Racah-Wigner normalization.

[‡]) Often called the generalized Jones polynomial.

2.6 The root of unity case

For applications most interesting is the case when q equals a primitive root of unity: $q^\ell = 1$. Then $\mathcal{U}_q(sl(n))$ possesses the real form $\mathcal{U}_q(sl(n, R))$. This case is rather complicated technically. One has to work with the restricted specialization $\mathcal{U}_q^{\text{res}}(sl(n))$ of $\mathcal{U}_q(sl(n))$ and the issue of the duality between the QF and QUE algebras becomes quite subtle. Fortunately, one can go on with the notion of the modular Hopf algebra [9].

DEFINITION 4: Consider a ribbon Hopf algebra $(\mathcal{A}, \check{R}, v)$ equipped with a distinguished family $\{V_j\}_{j \in \mathcal{S}}$ of irreducible \mathcal{A} -modules indexed by a finite set \mathcal{S} including the trivial representation V_0 . $(\mathcal{A}, \check{R}, v)$ is called a *modular Hopf algebra* if the following requirements are fulfilled:

1. $\text{qdim } V_j \neq 0, \forall j \in \mathcal{S}$.
2. The set $\{V_j\}_{j \in \mathcal{S}}$ is equipped with an involution $j \rightarrow j^*$ such that $V_{j^*} = V_j^*$ and $V_j^{**} = V_j$.
3. For any sequence $j_1, \dots, j_n \in \mathcal{S}$

$$V_{j_1} \otimes V_{j_2} \otimes \dots \otimes V_{j_n} = \bigoplus_{i \in \mathcal{S}} V_i^{\oplus m_i} \oplus I, \quad m_i \in \mathbb{N}$$

as \mathcal{A} -modules and for all \mathcal{A} -module endomorphisms, f , of the ideal I

$$\text{qtr}(f) = 0$$

4. Let s_{ij} be the quantum invariant of the Hopf link, two components of which are colored with irreps i and $j \in \mathcal{S}$

$$s_{ij} = \text{qtr} \left[\begin{array}{c} \text{---} \\ | \\ \bigcap \\ | \\ \text{---} \end{array} \begin{array}{c} i \\ j \end{array} \right]$$

then the matrix $(s_{ij})_{i,j \in \mathcal{S}}$ is invertible. \square

Let us take the row of the inverse matrix s^{-1} corresponding to the trivial representation V_0 , then

$$\sum_{j' \in \mathcal{S}} (s^{-1})_{0j'} s_{j'j} = \delta_{0,j} \tag{2.59}$$

We can consider Eq. (2.59) as an analog of the basic integral (2.51) with the obvious action of the co-multiplication:

$$\Delta \left(\begin{array}{c} \text{---} \\ | \\ \bigcap \\ | \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ | \\ \bigcap \\ | \\ \text{---} \end{array} \tag{2.60}$$

where N_k is the number of k -dimensional cells in a complex C ; $J_{j_1 \dots j_{N_2}}^{i_1 \dots i_{N_1}}(\mathcal{L})$ is the quantum invariant of a link \mathcal{L} given by a Heegaard diagram associated to the complex C . Let us denote $\omega = \sum_{i \in S} v_i d_i$.

THEOREM 1: $\mathcal{I}(\mathcal{M}) = \mathcal{Z}_0(C)/\omega^{N_0+N_3-2}$ is a topological invariant of a manifold \mathcal{M} represented by a complex C . $\mathcal{I}(\mathcal{M})$ is multiplicative with respect to the connected sum:

$$\mathcal{I}(\mathcal{M}) = \mathcal{I}(\mathcal{M}_1)\mathcal{I}(\mathcal{M}_2), \quad \text{if } \mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2$$

and $\mathcal{I}(S^3) = 1$.

PROOF The Heegaard splitting associated to a cell complex C having N_k cells in the k 'th dimension gives a handlebody, H_g , of the genus $g = N_1 - N_0 + 1$. Let us fix g independent α -cycles of the Heegaard diagram and take the corresponding integrals (*i.e.*, sums over i 's in Eq. (2.63)) firstly. By applying the CGC decomposition and then using the orthogonality (2.61), we deform the set of the characteristic curves in the link \mathcal{L} into some 3-valent ribbon graph \mathcal{G} . Every application of Eq. (2.61) destroys a handle of H_g . Therefore, having taken the g integrals, we obtain a spherical ribbon 3-valent graph \mathcal{G} plus a collection of $N_1 - g = N_0 - 1$ disjoint unlinked loops corresponding to the rest of the α -cycles. The integrals associated to them give ω^{N_0-1} . Now, we can recover the initial configuration of the characteristic curves[§] by restoring the g integrals corresponding to the independent α -cycles. In this way we obtain a cell decomposition of \mathcal{M} with only one 0-cell and every 1-cell corresponding to a generator of $\pi_1(\mathcal{M})$. This procedure is the direct analog of fixing an axial gauge in LGT.

The Heegaard splitting is obviously symmetric with respect to the Poincaré duality, therefore we can repeat the previous procedure with roles of the α -cycles and the characteristic curves interchanged. In this way we fix a set of g independent characteristic curves and pick up the factor ω^{N_3-1} . Thus, we finish with some balanced presentation of $\pi_1(\mathcal{M})$.

To prove the topological invariance, we have to show that $\mathcal{I}(\mathcal{M})$ is not changed [i] by Dehn twists on contractible loops, [ii] by the cycle slide and [iii] by the stabilization.

i) Invariance under the Dehn twists on loops contractible inside H_g is obvious. By taking the g integrals, we cut all handles and always get the same 3-valent ribbon graph \mathcal{G} .

ii) Invariance under the cycle slide follows from the analog of Haar measure invariance as illustrated in Figure 2.

iii) The stabilization consists in adding a handle to H_g and extending a gluing homomorphism h by the identity on its boundary. It amounts to the addition to \mathcal{L} of one α -cycle and one characteristic curve forming the Hopf link. Therefore, the integration associated to the new handle attaches the trivial representation to the new characteristic curve and it is unimportant how it is linked with the other α -cycles.

[§]) or another one equivalent to it.

$$\begin{aligned}
& \sum v_i \sum v_j \text{qtr} \left[\begin{array}{c} \text{Diagram 1: A box labeled } f \text{ with four vertical lines entering from the bottom and four exiting from the top. A loop labeled } j \text{ is on the left, and a loop labeled } i \text{ is on the right, both crossing the top lines.} \end{array} \right] = \sum v_i \sum v_j \text{qtr} \left[\begin{array}{c} \text{Diagram 2: Similar to Diagram 1, but the loops } j \text{ and } i \text{ are now on the top lines, and the bottom lines are straight.} \end{array} \right] = \\
& \frac{1}{d} \sum v_i \text{qtr} \left[\begin{array}{c} \text{Diagram 3: A box labeled } f \text{ with four vertical lines entering from the bottom and four exiting from the top. A loop labeled } i \text{ is on the right, and a loop labeled } j \text{ is on the left, both crossing the top lines.} \end{array} \right] = \sum v_i \sum v_j \text{qtr} \left[\begin{array}{c} \text{Diagram 4: Similar to Diagram 3, but the loops } j \text{ and } i \text{ are now on the top lines, and the bottom lines are straight.} \end{array} \right]
\end{aligned}$$

Figure 2: Invariance under the cycle slide.

To show the multiplicative nature of the invariant, let us choose such a cell decomposition of $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2$ that a sphere dividing \mathcal{M}_1 and \mathcal{M}_2 consists of only one 0-cell and one 2-cell. Then the corresponding characteristic curve is linked with no α -cycle and the corresponding link \mathcal{L} in Eq. (2.63) has two connected components.

The normalization $\mathcal{I}(S^3) = 1$ follows from the observation that the Hopf link corresponds to a genus 1 Heegaard splitting of the sphere. \square

Remarks: 1) The meaning of the choice of the Boltzmann weight coefficients made in Eq. (2.63) is clear. They corresponds to the δ -function weights. Therefore, $\mathcal{Z}_0(\mathcal{M})$ can be regarded as a generalization of Eq. (2.10). In contrast to the finite group partition function, the q -deformed model is obviously self-dual with respect to the Poincaré duality of 3-folds.

2) Let us consider a simplicial complex $C^{(s)}$. If we take in the expression (2.63) for the partition function $\mathcal{Z}_0(C^{(s)})$ all the sums associated to triangles in $C^{(s)}$ prior to the others, then the answer is identical to the definition of the Turaev-Viro invariant. Indeed, for each triangle we find the tangle equivalent to the product of two $3j$ -symbols:

$$\sum v_i \begin{array}{c} \text{Diagram: A box with three vertical lines entering from the bottom, labeled } j_1, j_2, j_3. A loop labeled } i \text{ is on the right, crossing the top lines.} \end{array} = \sum v_i \begin{array}{c} \text{Diagram: A box with three vertical lines entering from the bottom, labeled } j_1, j_2, j_3. A loop labeled } i \text{ is on the right, crossing the top lines.} \end{array} = \frac{1}{d_{j_3}} \begin{array}{c} \text{Diagram: A box with three vertical lines entering from the bottom, labeled } j_1, j_2, j_3. A loop labeled } i \text{ is on the right, crossing the top lines.} \end{array} \quad (2.64)$$

By closing all the tangles, we obtain a Racah-Wiegner $6j$ -symbol

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} = \frac{1}{d_{j_6} \sqrt{d_{j_2} d_{j_5}}} \text{ (diagram)} \quad (2.65)$$

inside each tetrahedron of the simplicial complex. The indices, j_1, \dots, j_{N_1} , are attached to 1-simplexes of $C^{(s)}$. Taking the sums over them we arrive at the Turaev-Viro state sum invariant [6]:

$$\mathcal{I}_0(C^{(s)}) = \omega^{N_1 - N_2 + 2} \sum_{\{j_k \in \mathcal{S}\}} \prod_{k=1}^{N_1} v_{j_k} \prod_{t=1}^{N_3} \left\{ \begin{array}{ccc} j_{t_1} & j_{t_2} & j_{t_3} \\ j_{t_4} & j_{t_5} & j_{t_6} \end{array} \right\} \quad (2.66)$$

where the 6-tuple (t_1, \dots, t_6) denotes six edges of the t -th tetrahedron. Explicit expressions for $(s^{-1})_{ij}$ in the sl_2 case are given in Ref. [9]. Thus, Eq. (2.63) can be regarded as a general definition of the Turaev-Viro invariant.

3) The expression for $\mathcal{Z}_0(\mathcal{M})$ given in Eq. (2.63) coincides with the Reshetikhin-Turaev construction of 3-fold invariants I_{RT} via the surgery representation [9]. Therefore, $\mathcal{Z}_0(\mathcal{M})$ is automatically invariant under the Kirby calculus applied formally to the link \mathcal{L} . It means that, given a manifold \mathcal{M} , there exists another one \mathcal{N} such that $\mathcal{I}(\mathcal{M}) = I_{RT}(\mathcal{N})$. As $\mathcal{I}(\mathcal{M}) = |I_{RT}(\mathcal{M})|^2$, we conclude that $\mathcal{N} \cong \mathcal{M} \# \overline{\mathcal{M}}$ ($\overline{\mathcal{M}}$ is \mathcal{M} with the opposite orientation). A simple illustration in the case of lens spaces can be found in Ref. [7].

2.8 Bounded manifolds and links

Every set of disjoint simple closed curves $\{\gamma\}$ on a handlebody H determines a bounded 3-manifold \mathcal{M} constructed by gluing plates to annular neighborhoods of the curves. It can be shown that every orientable bounded 3-manifold can be obtained in this way. The handlebody in this construction is a tubular neighborhood of a 1-skeleton of \mathcal{M} . Therefore we can straightforwardly apply the qQCD₃ functor in the bounded case. For it, we [i] fix a system of α -cycles on ∂H ; [ii] color curves from $\{\gamma\}$ with \mathcal{U}_q irreps; [iii] repeat steps 3, 4 and 5 from Definition 1 without any modification. It suggest the following interpretation of our construction. A *spine* is a 2-dimensional polyhedron which can be embedded in some 3-manifold. Any 3-manifold with a boundary collapses to a spine. Let us delete a ball from every 3-cell of a closed complex C . In such a way we obtain a bounded manifold which collapses to a 2-skeleton K_2 of C . If C is dual to a simplicial complex, K_2 is called a *standard spine*. Matveev has introduced two moves which relate all standard spines of the same manifold [19]. It can be easily shown that $\mathcal{I}(\mathcal{M})$ from the previous section is invariant under the Matveev moves.

The definition of the qQCD_3 functor uses an immersion of K_2 into R^3 and depends on it. It seems to be an intrinsic feature of q -deformed LGT rather than a defect of our presentation. Only gauge invariant singlet quantities (the partition function, for example) are independent of a way K_2 is immersed into R^3 .

One of the advantages of our presentation of qQCD_3 is a relative simplicity of introducing Wilson loops in it. In classical LGT, a loop average is defined as

$$A(L_1, \dots, L_m) = \frac{1}{\mathcal{Z}_\beta} \int_G \prod_\ell dg_\ell \prod_f W_\beta(h_{\partial f}) \prod_{i=1}^m \text{tr}_{V_{j_i}}[h_{L_i}] \quad (2.67)$$

where $\{L\}$ are m closed curves embedded into a 1-skeleton K^1 of a complex C . We color the i -th curve with a representation j_i of a gauge group G . The holonomy h_L is defined in Eq. (2.2).

In the q -deformed case, we have to specify a link formed by the collection of curves in a manifold $\mathcal{M} \cong C$. For it we represent the curves $\{L\}$ by a set of disjoint ribbon loops on a boundary $\partial H'$ of a handlebody $H' \subset H$ (as usual, H' and H are tubular neighborhoods of K^1 and H' lies inside H : $H \cap H' = H'$, $H \cup H' = H$). If it is not possible, then one has to take a finer subdivision of \mathcal{M} . In the case of links in R^3 , it is a standard technical trick to realize a link as a system of disjoint loops on a handlebody embedded into R^3 . And we simply use it as a definition.

We can apply the qQCD_3 functor to such a composite handlebody without any additional modification. Loops from $\{L\}$ enters on equal footing with characteristic curves. One can repeat the same argument as in the partition function case to prove that an answer is independent of an embedding of H in R^3 . However, it does not mean that the q -deformation of Eq. (2.67) gives no non-trivial knot invariant. Let us consider a link in R^3 . There has to exist a trivial embedding such that characteristic curves of a Heegaard diagram lying on ∂H are unlinked and contractible in $R^3 \setminus H$. Therefore the sums over their colors disjoint α -cycles on ∂H and the link of curves $\{L\}$ on $\partial H'$ (in other words, cut handles of the complementary handlebody $S^3 \setminus H$). What remains is exactly the Jones polynomial associated to the link $\{L\}$.

The comprehensive treatment of quantum invariants of links and 3-valent graphs in 3-manifolds can be found in Ref. [20].

3 qQCD_2

We define qQCD_2 functor by applying the qQCD_3 one to an embedding of an oriented 2-manifold M_g^2 in R^3 . In the topological limit, we can consider the simplest cell decomposition of M_g^2 consisting of a single 2-cell and $2g$ 1-cells. A tubular neighborhood of its 1-skeleton is a handlebody H of the genus $2g$. The Heegaard diagram has only one characteristic curve. Each integral destroys a handle of H and contributes a factor $1/d_j$ to an

answer. The calculation is reduced to a repeated application of Eq. (2.61) and one easily gets

$$\mathcal{I}(M_g^2) = \omega^{2g-1} \mathcal{Z}_0(M_g^2) = \omega^{2g-1} \sum_{j \in \mathcal{S}} v_j Z_j = \sum_{j \in \mathcal{S}} v_j \left(\frac{d_j}{\omega} \right)^{1-2g} \quad (3.1)$$

Let us consider a concrete example of the quantum group $\mathcal{U}_q(sl(2))$, at $q = e^{\frac{2\pi i}{k+2}}$. In this case, the set of modules in the definition of the modular Hopf algebra is given by the fusion ring V_j ($j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{k}{2}$) and $d_j = \frac{\sin(\frac{2j+1}{k+2}\pi)}{\sin \frac{\pi}{k+2}}$; $\omega = \frac{\sqrt{(k+2)/2}}{\sin \frac{\pi}{k+2}}$; $v_j = d_j/\omega$. We find

$$\mathcal{I}(M_g^2) = \sum_{j=0, \frac{1}{2}, \dots, \frac{k}{2}}^{k/2} \left(\frac{2 \sin^2 \left(\frac{2j+1}{k+2} \pi \right)}{k+2} \right)^{1-g} \quad (3.2)$$

These are known as Verlinde's numbers. They are all integer and equal to the dimensions of spaces of conformal blocks in the WZW model on a genus g Riemann surface.

If one starts with a more complicated cell decomposition of a Riemann surface, then one has simply apply the orthogonality relation (2.61) till all handles of H are destroyed.

In two dimensions, local properties of qQCD₂ functor can be formalized in a pure algebraic way. For it, let us cut from M_g^2 a piece which can be projected on a plane R^2 . It gives a subdivision (triangulation, say) of some region on the plane. There is a natural cyclic order of edges incident to a vertex. Following Fock and Rosly [15], one introduces a *ciliation* at every vertex, *i.e.*, breaks this order. Let us say that an edge ℓ_1 goes after ℓ_2 ($\ell_1 > \ell_2$), if an anti-clockwise angle $\varphi(\ell_1)$ between the edge ℓ_1 and the x -axis is bigger than an angle $\varphi(\ell_2)$ between ℓ_2 and the x -axis: $\varphi(\ell_1) > \varphi(\ell_2)$. We assume that no edge is parallel to the x -axis, and orient edges in the y -direction. Say, put an arrow at an end having a bigger y coordinate. Assuming that any two vertices are connected at most by one edge, we can numerate edges by ordered pairs of vertices $(i, j) \cong \nearrow_j^i$.

Alekseev, Grosse and Schomerus have introduced the following algebra of gauge fields $U_{(i,j)}$ [13]:

i) If two edges have no common vertices, fields are co-commutative:

$$\overset{1}{U}_{(i,j)} \overset{2}{U}_{(n,m)} = \overset{2}{U}_{(n,m)} \overset{1}{U}_{(i,j)}$$

here i, j, n and m are all distinct.

ii) If two edges share a vertex, then

$$\overset{1}{U}_{(i,j)} \overset{2}{U}_{(k,j)} = \begin{cases} \overset{2}{U}_{(k,j)} \overset{1}{U}_{(i,j)} \check{R}_{12} & \text{if } (i, j) > (k, j) \\ \overset{2}{U}_{(k,j)} \overset{1}{U}_{(i,j)} \check{R}_{12}^{-1} & \text{if } (i, j) < (k, j) \end{cases}$$

We picture these relations as

$$\begin{array}{c}
 \begin{array}{ccc}
 i & k & i \quad k \\
 \vdots & \vdots & \diagdown \quad \diagup \\
 \boxed{a} & \boxed{b} & = \boxed{b} \quad \boxed{a} \\
 | & | & \diagup \quad \diagdown \\
 & j & j
 \end{array}
 \end{array}
 \quad \text{if } (i, j) > (k, j)$$

$$\begin{array}{c}
 \begin{array}{ccc}
 k & i & k \quad i \\
 \diagdown \quad \diagup & & \vdots \quad \vdots \\
 \boxed{a} & \boxed{b} & = \boxed{b} \quad \boxed{a} \\
 | & | & \diagup \quad \diagdown \\
 & j & j
 \end{array}
 \end{array}
 \quad \text{if } (i, j) < (k, j)$$

We have drawn in solid lines \mathcal{U}_q -elements figuring in the AGS relations associated to the j -th vertex. All attached to other vertices are dashed.

iii) Fields attached to the same edge form a quasi-triangular Hopf algebra:

1. $R_{12}^{-1} \overset{1}{U}_{(i,j)} \overset{2}{U}_{(i,j)} R_{12} = \overset{2}{U}_{(i,j)} \overset{1}{U}_{(i,j)}$
2. $\cdot \overset{1}{U}_{(j,i)} \overset{2}{U}_{(i,j)} = 1$. This property is sometimes called the cancellation of a back-tracking:

$$\begin{array}{c}
 i \\
 \diagup \quad \diagdown \\
 \boxed{a} \quad \boxed{a} \\
 | \\
 j
 \end{array}
 = \begin{array}{c} | \end{array}$$

Remarks: 1) The properties [i] and [iii] are obviously agreed with our definition of qQCD_2 (see, *e.g.*, , the pictorial illustration in Figure 1, and the discussion preceding it). The relations [ii] follow from transformations of modules associated to vertex tangles. Of course, being made, such a move has to be compensated somewhere by its reciprocal for a whole construction to remain invariant.

2) As a closed surface can be projected onto R^2 only locally, one has to use gluing homomorphisms to assembly a Riemann surface out of flat pieces. These homomorphisms match the gauge field algebra relations on different pieces and have to be added in order to complete the construction.

3) The set of the AGS relations is distinguished by an observation that they generate a lattice Kac-Moody algebra in the sense of Ref. [21]. However, they do not constitute all possible symmetries of the qQCD_2 functor.

4 Concluding remarks

1. The first natural question to ask is whether the results of this paper could be generalized to higher dimensions. The answer is certainly “No”! The reason for it is that, in dimensions bigger than 3, there is no natural ordering of faces incident to an edge in a complex. It restricts the class of acceptable Hopf algebras to triangular ones. Then the corresponding construction essentially coincides with the classical Wilsonian LGT.
2. It is tempting to interpret the topological invariant considered in this paper as some suitable generalization of Eq. (2.10) and the construction of topological qCDQ₃ as a generalization of $H^1(C, G)$. Unfortunately, we are able to say nothing constructive about it. However, in the 2-dimensional case, a notion of a quantum moduli space could presumably be formulated [15], which leaves some hope for the future.
3. We conjecture that qQCD₃ with $\mathcal{U}_q(su(n))$ gauge group possesses a continuum limit equivalent to a gauge theory whose action includes both Yang-Mills and Chern-Simons terms. One could introduce a non-zero coupling constant in the root-of-unity case as well, which implies some deformation of Chern-Simons theory. A meaning of this procedure is absolutely unclear to us.

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